Digital Combining-Weight Estimation for Broadband Sources Using Maximum-Likelihood Estimates

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The algorithm described in [1] for estimating the optimum combining weights for the Ka-band (33.7-GHz) array feed compensation system is compared with the maximum-likelihood estimate. This provides some improvement in performance, with an increase in computational complexity. However, the maximum-likelihood algorithm is simple enough to allow implementation on a PC-based combining system.

I. Introduction

We consider the problem of estimating combining weights for a signal received by an antenna array. The signal is modeled as a Gaussian random variable, and independent Gaussian noise is added in each channel. An estimation method that has been proposed is treated in [1]. Here we compare that method with the method that uses maximum-likelihood (ML) estimates of the pertinent parameters. The computations required for these estimates, while more complex than the computations of [1], are well within the capabilities of a small on-site computer.

II. The Maximum-Likelihood Equations

The received signal in the kth channel at time i is assumed to be

$$\tilde{r}_k(i) = \tilde{S}_k \tilde{a}(i) + \tilde{n}_k(i), \qquad k = 1, 2, \ldots, K \qquad (1)$$

where the $\tilde{n}_k(i)$ and $\tilde{a}(i)$ are independent complex Gaussian random variables, $\tilde{n}_k(i)$ is $N(0, 2\sigma_k^2)$, and $\tilde{a}(i)$ is $N(0, 1)$. Then[1]

$$\tilde{c}_{jk} = E\left(\overline{\tilde{r}_j(i)}\tilde{r}_k(i)\right) = 2\sigma_k^2 \delta_{jk} + \overline{\tilde{S}_j}\tilde{S}_k \qquad (2)$$

If $\tilde{\mathbf{C}}$ is the complex $K \times K$ matrix with entries $\tilde{c}_{jk}$, then the real and imaginary parts of the $\tilde{r}_k(i)$ (for $k = 1, \ldots, K$, i fixed) have a $2K$-dimensional distribution with density[2]

$$p_i = \frac{1}{\pi^K \det(\tilde{\mathbf{C}})} \exp\left[-\sum_{j,k=1}^{K} \tilde{r}_j(i)(\tilde{\mathbf{C}}^{-1})_{jk}\overline{\tilde{r}_k(i)}\right] \qquad (3)$$

[1] The overbar denotes a complex conjugate.

[2] The arguments $\tilde{r}_1(i), \ldots, \tilde{r}_K(i)$ of p_i are not shown explicitly.

It is convenient to introduce the quantities

$$\tilde{T}_k = \frac{\tilde{S}_k}{\sqrt{2}\sigma_k} \tag{4}$$

Then

$$\tilde{c}_{jk} = 2\sigma_j\sigma_k\left(\delta_{jk} + \overline{\tilde{T}_j}\tilde{T}_k\right) \tag{5}$$

and the elements of the inverse matrix are

$$\tilde{b}_{jk} = (\tilde{\mathbf{C}}^{-1})_{jk} = \frac{1}{2\sigma_j\sigma_k}\left(\delta_{jk} - \frac{1}{\gamma}\overline{\tilde{T}_j}\tilde{T}_k\right) \tag{6}$$

where

$$\gamma = 1 + \sum_{k=1}^{K}|\tilde{T}_k|^2 \tag{7}$$

Also, we have

$$\det(\tilde{\mathbf{C}}) = 2^K\left(\prod_{k=1}^{K}\sigma_k^2\right)\gamma \tag{8}$$

Using these values in Eq. (3),

$$p_i = \frac{1}{(2\pi)^K\left(\prod_{k=1}^{K}\sigma_k^2\right)\gamma} \times \exp\left[-\frac{1}{2}\sum_{j,k=1}^{K}\frac{1}{\sigma_j\sigma_k}\tilde{r}_j(i)\left(\delta_{jk} - \frac{1}{\gamma}\overline{\tilde{T}_j}\tilde{T}_k\right)\overline{\tilde{r}_k(i)}\right] \tag{9}$$

We define the likelihood function as

$$\Lambda = \frac{1}{L}\sum_{i=1}^{L}\ln p_i \tag{10}$$

In terms of the sample covariances

$$\tilde{a}_{jk} = \frac{1}{L}\sum_{i=1}^{L}\overline{\tilde{r}_j(i)}\tilde{r}_k(i) \tag{11}$$

It follows from Eq. (9) that

$$\Lambda = -\sum_{k=1}^{K}\ln(2\pi\sigma_k^2) - \ln(\gamma) - \frac{1}{2}\sum_{j,k=1}^{K}\frac{1}{\sigma_j\sigma_k}\tilde{a}_{kj}\left(\delta_{jk} - \frac{1}{\gamma}\overline{\tilde{T}_j}\tilde{T}_k\right) \tag{12}$$

III. Cascaded Maximum-Likelihood Estimates

For maximum-likelihood estimates $\hat{\tilde{S}}_k$ and $\hat{\sigma}_k$, or equivalently $\hat{\tilde{T}}_k$ and $\hat{\sigma}_k$, we need to solve the equations obtained by setting the derivatives of Λ equal to zero. This system of equations must be solved iteratively. It need not have a unique solution, for the parameters are not even determined by the statistics of the signals [Eq. (1)] unless at least three of the $\tilde{S}_k$ are nonzero. For this reason, this approach is not pursued here. We assume that the σ_k are estimated from separate observations with the antenna pointed "off source." These noise estimates are themselves maximum-likelihood estimates obtained from the noise samples by differentiating Eq. (12) with respect to σ_j (assuming $\tilde{T}_k = 0$):

$$\hat{\sigma}_j = \sqrt{\frac{1}{2}\tilde{a}_{jj}}$$

where $\tilde{a}_{jj}$ is given by Eq. (11). The maximum-likelihood estimates of the $\tilde{T}_k$ use these noise estimates.

Differentiating Eq. (12) with respect to $\overline{\tilde{T}_j}$, we get

$$\left(-\frac{1}{\gamma} - \frac{1}{2\gamma^2}\sum_{k,m=1}^{K}\frac{1}{\sigma_k\sigma_m}\tilde{a}_{km}\overline{\tilde{T}_m}\tilde{T}_k\right)\tilde{T}_j + \frac{1}{2\gamma}\sum_{k=1}^{K}\frac{1}{\sigma_j\sigma_k}\tilde{a}_{kj}\tilde{T}_k = 0$$

It can be shown that this is equivalent to the simpler condition

$$\sum_{k=1}^{K} \frac{\tilde{a}_{kj}}{\sigma_k \sigma_j} \tilde{T}_k = 2\gamma \tilde{T}_j$$

Replacing the parameters by their estimates, we get

$$\sum_{k=1}^{K} \frac{\tilde{a}_{kj}}{\hat{\sigma}_k \hat{\sigma}_j} \hat{\tilde{T}}_k = 2\hat{\gamma} \hat{\tilde{T}}_j \tag{13}$$

This equation states that the complex K vector with components $\hat{\tilde{T}}_k$ is an eigenvector of the matrix composed of the elements $\tilde{a}_{kj}/(\hat{\sigma}_k \hat{\sigma}_j)$, with eigenvalue $2\hat{\gamma}$. If we replace the matrix elements by their mean values, using the true values of the σ_k, then this matrix has $K-1$ eigenvalues equal to 2, and one larger eigenvalue 2γ, corresponding to the eigenvector $\tilde{T}_k$. Hence, the estimates $\hat{\tilde{T}}_k$ can be found in terms of the σ_k by numerically finding the largest eigenvalue of the matrix in Eq. (13) and its eigenvector. The eigenvector must be scaled so that the eigenvalue $2\hat{\gamma}$ satisfies the relation [from Eq. (7)]

$$\hat{\gamma} = 1 + \sum_{k=1}^{K} |\hat{\tilde{T}}_k|^2 \tag{14}$$

The method for solving Eq. (13) is described briefly in Appendix A.

IV. Variance of the Cascaded ML Estimates for the $\tilde{T}_j$

For a large L, the sample covariances $\tilde{a}_{jk}$ are close to their mean values:

$$\tilde{a}_{jk} = \tilde{c}_{jk} + \delta \tilde{a}_{jk} \tag{15}$$

where the difference $\delta \tilde{a}_{jk}$ has a mean of zero and a small variance. It is easily shown that

$$E(\overline{\delta \tilde{a}_{jk}} \delta \tilde{a}_{lm}) = \frac{4}{L} \sigma_j \sigma_k \sigma_l \sigma_m$$

$$\times \left(\delta_{km} + \overline{\tilde{T}_k} \tilde{T}_m\right) \left(\delta_{lj} + \overline{\tilde{T}_l} \tilde{T}_j\right) \tag{16}$$

If the estimates $\hat{\sigma}_j$ are close to the correct values,

$$\hat{\sigma}_j = \sigma_j + \delta \sigma_j \tag{17}$$

Then the estimates $\hat{\tilde{T}}_j$ are close to the true values,

$$\hat{\tilde{T}}_j = \tilde{T}_j + \delta \tilde{T}_j \tag{18}$$

Expand Eq. (15), keeping only those terms which are of first order in the deviations $\delta \tilde{a}_{jk}$, $\delta \sigma_j$, and $\delta \tilde{T}_j$. We get

$$\sum_{k=1}^{K} \left[\frac{\delta \tilde{a}_{kj}}{2\sigma_k \sigma_j} \tilde{T}_k + \frac{\tilde{c}_{kj}}{2\sigma_k \sigma_j} \delta \tilde{T}_k - \frac{\tilde{c}_{kj}}{2\sigma_k \sigma_j} \tilde{T}_k \left(\frac{\delta \sigma_j}{\sigma_j} + \frac{\delta \sigma_k}{\sigma_k} \right) \right] =$$

$$\gamma \delta \tilde{T}_j + \tilde{T}_j \sum_{k=1}^{K} \left(\overline{\tilde{T}}_k \delta \tilde{T}_k + \tilde{T}_k \overline{\delta \tilde{T}_k} \right)$$

Using the formula in Eq. (5) and simplifying, the result is

$$(\gamma - 1) \delta \tilde{T}_j + \tilde{T}_j \sum_{k=1}^{K} \tilde{T}_k \overline{\delta \tilde{T}_k} =$$

$$\sum_{k=1}^{K} \frac{\delta \tilde{a}_{kj}}{2\sigma_k \sigma_j} \tilde{T}_k - \tilde{T}_j \sum_{k=1}^{K} \frac{\delta \sigma_k}{\sigma_k} |\tilde{T}_k|^2$$

$$- (\gamma + 1) \tilde{T}_j \frac{\delta \sigma_j}{\sigma_j} \tag{19}$$

If we multiply this equation by $\overline{\delta \tilde{T}_j}$ and sum over j, we get

$$(\gamma - 1) \sum_{k=1}^{K} \left(\overline{\tilde{T}}_k \delta \tilde{T}_k + \tilde{T}_k \overline{\delta \tilde{T}_k} \right) =$$

$$\sum_{j,k=1}^{K} \frac{\delta \tilde{a}_{kj}}{2\sigma_k \sigma_j} \overline{\tilde{T}}_j \tilde{T}_k - 2\gamma \sum_{j=1}^{K} \frac{\delta \sigma_j}{\sigma_j} |\tilde{T}_j|^2 \tag{20}$$

This equation determines the real part of $\sum \overline{\tilde{T}}_k \delta \tilde{T}_k$. The imaginary part of this sum is undetermined, since the $\tilde{T}_k$ can be multiplied by an arbitrary common complex factor of absolute value 1. From Eq. (20), we can set

$$\sum_{k=1}^{K} \overline{\tilde{T}}_k \delta \tilde{T}_k = \frac{1}{2(\gamma-1)} \sum_{m,k=1}^{K} \frac{\delta \tilde{a}_{km}}{2\sigma_k \sigma_m} \overline{\tilde{T}}_m \tilde{T}_k$$

$$- \frac{\gamma}{\gamma-1} \sum_{k=1}^{K} \frac{\delta \sigma_k}{\sigma_k} |\tilde{T}_k|^2 + jw \qquad (21)$$

where w is a real quantity not yet specified. Using this expression in Eq. (19), we get

$$(\gamma-1)\delta \tilde{T}_j = \sum_{k=1}^{K} \frac{\delta \tilde{a}_{kj}}{2\sigma_k \sigma_j} \tilde{T}_k - \frac{\tilde{T}_j}{2(\gamma-1)}$$

$$\times \sum_{m,k=1}^{K} \frac{\delta \tilde{a}_{km}}{2\sigma_k \sigma_m} \overline{\tilde{T}}_m \tilde{T}_k$$

$$+ \frac{\tilde{T}_j}{\gamma-1} \sum_{k=1}^{K} \frac{\delta \sigma_k}{\sigma_k} |\tilde{T}_k|^2$$

$$- (\gamma+1) \frac{\delta \sigma_j}{\sigma_j} \tilde{T}_j - jw\tilde{T}_j \qquad (22)$$

In the following, we will take $w = 0$, since this turns out to give the best results. The estimate $\hat{\sigma}_j$, found with the signal absent, is

$$\hat{\sigma}_j = \sqrt{\frac{1}{2} \tilde{a}_{jj}}$$

Expanding as above,

$$\frac{\delta \sigma_j}{\sigma_j} = \frac{\delta \tilde{a}_{jj}}{4\sigma_j^2}$$

Using Eq. (16) (with no signal), we get

$$E\left(\frac{\delta \sigma_j \delta \sigma_k}{\sigma_j \sigma_k}\right) = \frac{1}{4M} \delta_{jk} \qquad (23)$$

where M is the number of samples used in the noise estimates.

Square Eq. (22) and take the expected value. We denote the deviation of Eq. (22) by the prefix δ_0 to distinguish it from that obtained by other methods below. Using Eqs. (16) and (23), the result is

$$E(|\delta_0 \tilde{T}_j|^2) = \frac{1}{(\gamma-1)^2} \left\{ \frac{1}{L} \gamma \left[\gamma - 1 - |\tilde{T}_j|^2 + \frac{1}{4} \gamma |\tilde{T}_j|^2 \right] \right.$$

$$+ \frac{1}{4M} \left[\frac{|\tilde{T}_j|^2}{(\gamma-1)^2} \sum_{k=1}^{K} |\tilde{T}_k|^4 - 2 \frac{\gamma+1}{\gamma-1} |\tilde{T}_j|^4 \right.$$

$$\left. \left. + (\gamma+1)^2 |\tilde{T}_j|^2 \right] \right\} \qquad (24)$$

The sum over j of this expression leads to

$$\sum_{j=1}^{K} E(|\delta_0 \tilde{T}_j|^2) =$$

$$\frac{1}{\gamma-1} \left\{ \frac{1}{L} \gamma \left[K - 1 + \frac{1}{4} \gamma \right] + \frac{1}{4M} \right.$$

$$\left. \times \left[(\gamma+1)^2 - \frac{2\gamma+1}{(\gamma-1)^2} \sum_{k=1}^{K} |\tilde{T}_k|^4 \right] \right\} \qquad (25)$$

Now we consider another estimate, where the first channel is taken to be the one with maximum signal strength, and $\tilde{S}_1$ (or $\tilde{T}_1$) is estimated first. By Eq. (5),

$$E(\tilde{a}_{11}) = 2\sigma_1^2 (1 + |\tilde{T}_1|^2)$$

This leads to the estimate

$$|\hat{\tilde{T}}_1|^2 = \frac{\tilde{a}_{11}}{2\hat{\sigma}_1^2} - 1 \qquad (26)$$

The resulting error $\delta_2 \tilde{T}_1$ has variance

$$E(|\delta_2 \tilde{T}_1|^2) = \left(\frac{1}{L} + \frac{1}{M} \right) \frac{(1 + |\tilde{T}_1|^2)^2}{4|\tilde{T}_1|^2} \qquad (27)$$

This estimate can be used to obtain an estimate for $\tilde{T}_j$ for $j \geq 2$ by using

$$\hat{\hat{\tilde{T}}}_j = \frac{\tilde{a}_{1j}}{2\hat{\sigma}_1\hat{\sigma}_j\hat{\hat{\tilde{T}}}_1} \qquad (j \geq 2) \tag{28}$$

The variance of this estimate is

$$E(|\delta_2\tilde{T}_j|^2) =$$

$$\frac{1}{4L|\tilde{T}_1|^4}\left[|\tilde{T}_j|^2(1+|\tilde{T}_1|^2)^2 + 4|\tilde{T}_1|^2(1+|\tilde{T}_1|^2)\right]$$

$$+ \frac{|\tilde{T}_j|^2}{4M|\tilde{T}_1|^4}(1+|\tilde{T}_1|^4) \tag{29}$$

V. Variance of the Combining-Weight Estimates

We now consider estimates for the weights

$$\tilde{w}_j = \frac{\overline{\tilde{S}_j}}{2\tilde{\sigma}_j^2} = \frac{\overline{\tilde{T}_j}}{\sqrt{2}\tilde{\sigma}_j}$$

obtained from the cascaded ML estimates of the signal-to-noise ratio (SNR) and noise parameters. The estimate

$$\hat{\tilde{w}}_j = \frac{\overline{\hat{\hat{S}}_j}}{2\hat{\tilde{\sigma}}_j^2} = \frac{\overline{\hat{\hat{T}}_j}}{\sqrt{2}\hat{\tilde{\sigma}}_j} \tag{30}$$

has a deviation $\delta\tilde{w}_j$ from the true value, given to the first order by

$$\delta\tilde{w}_j = \frac{1}{\sqrt{2}\sigma_j}\left(\overline{\delta\tilde{T}_j} - \overline{\tilde{T}_j}\frac{\delta\sigma_j}{\sigma_j}\right)$$

The mean-square value of this deviation can be computed for each of the estimation methods under consideration. We find

$$E(|\delta_0\tilde{w}_j|^2) = \frac{1}{2(\gamma-1)^2\sigma_j^2}\left\{\frac{1}{L}\gamma\left[\gamma - 1 - |\tilde{T}_j|^2 + \frac{1}{4}\gamma|\tilde{T}_j|^2\right]\right.$$

$$\left. + \frac{1}{4M}\left[\frac{|\tilde{T}_j|^2}{(\gamma-1)^2}\sum_{k=1}^{K}|\tilde{T}_k|^4 - \frac{4\gamma}{\gamma-1}|\tilde{T}_j|^4 + 4\gamma^2|\tilde{T}_j|^2\right]\right\} \tag{31}$$

$$E(|\delta_2\tilde{w}_j|^2) = \frac{1}{8L\sigma_j^2|\tilde{T}_1|^4}[|\tilde{T}_j|^2(1+|\tilde{T}_1|^2)^2 + 4(1-\delta_{j1})$$

$$\times |\tilde{T}_1|^2(1+|\tilde{T}_1|^2)] + \frac{|\tilde{T}_j|^2}{8M\sigma_j^2|\tilde{T}_1|^4}$$

$$\times (1 + 4\delta_{j1}|\tilde{T}_1|^2 + 4|\tilde{T}_1|^4) \tag{32a}$$

A modified form of this last method was considered in [1]. There, $\tilde{T}_1$ was estimated from Eq. (26) based on N samples, and the other $\tilde{T}_j$ were estimated by Eq. (28) for a later set of L samples. For this method, it can be shown that

$$E(|\delta\tilde{w}_j|^2) = \frac{|\tilde{T}_j|^2(1+|\tilde{T}_1|^2)^2}{8N\sigma_j^2|\tilde{T}_1|^4} + (1-\delta_{j1})$$

$$\times \frac{(1+|\tilde{T}_1|^2)(1+|\tilde{T}_j|^2)}{2L\sigma_j^2|\tilde{T}_1|^2} + \frac{|\tilde{T}_j|^2}{8M\sigma_j^2|\tilde{T}_1|^4}$$

$$\times (1 + 4\delta_{j1}|\tilde{T}_1|^2 + 4|\tilde{T}_1|^4) \tag{32b}$$

These values, divided by $|\tilde{w}_j|^2$, are plotted in Fig. 1. The values $|\tilde{T}_1|^2 = 0.05$ and $|\tilde{T}_j|^2 = 0.005$, $j \geq 2$, were used. M was fixed at the value 100,000, and $N = L$. There is no observable difference between the curves based on Eqs. (32a) and(32b). For a small L, the mean-square error from the maximum-likelihood formula is lower by 2.2 dB. For a large L, the other methods are better, but this is in an impractical range of the parameters. M should be at least as large as L, since the M samples provide noise estimates on which the subsequent estimates are based. The failure of the maximum-likelihood method in this range, which was applied only for estimating the $\tilde{T}_j$, shows that these estimates can be more strongly affected by errors in the noise estimates.

Figure 2 shows the maximum-likelihood curve for various values of M. These curves have the same general appearance as the corresponding curves for the method given in [1]. Again, on the right side of the figure, where the curves for various M are widely separated, the errors shown are higher than those shown in [1]. The significant points on these curves, with $M \geq L$, show an improvement over [1].

VI. Joint Maximum-Likelihood Estimates for the Equal Noise Case

An important special case which is seen in practice has all the σ_j equal. Here maximum-likelihood estimates can be used to simultaneously determine the noise and signal levels (the problems cited above no longer apply).

Denote the common value of the σ_j by σ. The formula in Eq. (12) for Λ becomes

$$\Lambda = -K \ln (2\pi\sigma^2) - \ln (\gamma)$$

$$- \sum_{j,k=1}^{K} \frac{\tilde{a}_{kj}}{2\sigma^2} \left(\delta_{jk} - \frac{1}{\gamma} \bar{\tilde{T}}_j \tilde{T}_k \right) \tag{33}$$

To find the noise and signal amplitudes simultaneously by the maximum-likelihood method, set the derivative of Λ with respect to σ equal to zero, and solve this equation together with Eq. (13). Simplifying the derivative by using Eq. (13), the equation obtained is

$$2\hat{\sigma}^2 (K - 1 + \hat{\gamma}) = \sum_{k=1}^{K} \tilde{a}_{kk} \tag{34}$$

Eliminating $\hat{\sigma}$ from Eq. (13), we get

$$\sum_{k=1}^{K} \tilde{a}_{kj} \hat{\tilde{T}}_k = \frac{\hat{\gamma}}{K - 1 + \hat{\gamma}} \left(\sum_{k=1}^{K} \tilde{a}_{kk} \right) \hat{\tilde{T}}_j \tag{35}$$

As before, this equation is solved by taking the vector with components $\hat{\tilde{T}}_j$ to be the eigenvector of the matrix $(\tilde{a}_{kj})$ corresponding to the largest eigenvalue. This eigenvalue must be the coefficient of $\hat{\tilde{T}}_j$ on the right, which determines $\hat{\gamma}$, and hence determines the $\hat{\tilde{T}}_j$ up to a common complex factor of absolute value 1. The noise estimate is then given by Eq. (34).

To get variance estimates, proceed as before. The equation analogous to Eq. (22) is

$$(\gamma - 1)\delta\tilde{T}_j = \sum_{k=1}^{K} \frac{\delta\tilde{a}_{kj}}{2\sigma^2} \tilde{T}_k - \frac{K - 1 - \gamma}{2(K-1)(\gamma - 1)} \tilde{T}_j$$

$$\times \sum_{m,k=1}^{K} \frac{\delta\tilde{a}_{km}}{2\sigma^2} \bar{\tilde{T}}_m \tilde{T}_k - \frac{\gamma\tilde{T}_j}{2(K-1)} \sum_{k=1}^{K} \frac{\delta\tilde{a}_{kk}}{2\sigma^2} - jw\tilde{T}_j \tag{36}$$

where w is again an undetermined real quantity which will be set equal to 0. The noise error $\delta\sigma$ can be found from Eq. (34). Eliminating $\delta\tilde{T}_j$ by the use of Eq. (36), the result is

$$\frac{\delta\sigma}{\sigma} = \frac{1}{2(K-1)}$$

$$\times \left(\sum_{k=1}^{K} \frac{\delta\tilde{a}_{kk}}{2\sigma^2} - \frac{1}{\gamma - 1} \sum_{m,k=1}^{K} \frac{\delta\tilde{a}_{km}}{2\sigma^2} \bar{\tilde{T}}_m \tilde{T}_k \right) \tag{37}$$

To get variance estimates, square the expressions in Eqs. (36) and (37) and take the expected value. Denoting the deviation in $\tilde{T}_j$ by the prefix δ_4, we get

$$E(|\delta_4 \tilde{T}_j|^2) =$$

$$\frac{1}{(\gamma - 1)^2} \left\{ \frac{1}{L} \gamma \left[\gamma - 1 - |\tilde{T}_j|^2 + \frac{1}{4} \frac{K}{K-1} \gamma |\tilde{T}_j|^2 \right] \right\} \tag{38}$$

$$E\left(\frac{\delta\sigma^2}{\sigma^2} \right) = \frac{1}{4L(K-1)} \tag{39}$$

Summing Eq. (38),

$$\sum_{j=1}^{K} E(|\delta_4 \tilde{T}_j|^2) = \frac{\gamma}{L(\gamma - 1)} \left[K - 1 + \frac{1}{4} \frac{K}{K-1} \gamma \right] \tag{40}$$

Using the above formulas with Eq. (30),

$$E(|\delta_4 \tilde{w}_j|^2) = \frac{1}{2\sigma^2(\gamma - 1)^2}$$

$$\times \left[\gamma(\gamma - 1 - |\tilde{T}_j|^2) + \frac{1}{4}\gamma^2 |\tilde{T}_j|^2 \right.$$

$$\left. + \frac{(2\gamma - 1)^2}{4(K-1)} |\tilde{T}_j|^2 \right] \tag{41}$$

VII. Performance of the Weight Estimates

We now consider the combined signal, formed by taking the weighted sum

$$\tilde{z}(i) = \sum_{k=1}^{K} \hat{\tilde{w}}_k \tilde{r}_k(i) \tag{42}$$

where the weights are based on data before time i. If we use $\hat{\tilde{w}}_k = \tilde{w}_k + \delta\tilde{w}_k$ and express $\tilde{r}_k(i)$ by Eq. (1), we get four terms:

$$\tilde{z}(i) = \sum_{k=1}^{K} \tilde{w}_k \tilde{S}_k \tilde{a}(i) + \sum_{k=1}^{K} \tilde{w}_k \tilde{n}_k(i)$$

$$+ \sum_{k=1}^{K} \tilde{S}_k \tilde{a}(i) \delta\tilde{w}_k + \sum_{k=1}^{K} \tilde{n}_k(i) \delta\tilde{w}_k$$

$$= \sum_{k=1}^{K} \tilde{w}_k \tilde{S}_k \tilde{a}(i) + E_1 + E_2 + E_3 \tag{43}$$

The first term is the desired signal. The other terms are contributions to the error which can be considered separately, since their cross products have zero expectation.

Averaging over values of the current signal first, we have

$$E(|E_1|^2) = \sum_{k=1}^{K} 2\sigma_k^2 |\tilde{w}_k|^2 = \gamma - 1 \tag{44}$$

$$E(|E_2|^2) = \frac{1}{2} E\left(\left| \sum_{k=1}^{K} 2\sigma_k \tilde{T}_k \delta\tilde{w}_k \right|^2 \right) \tag{45}$$

$$E(|E_3|^2) = E\left(\sum_{k=1}^{K} 2\sigma_k^2 |\delta\tilde{w}_k|^2 \right) \tag{46}$$

These quantities are easily evaluated by using previous formulas.

The following estimation methods are referred to in Figs. 3 and 4:

(1) Method 1: the maximum-likelihood method of Section I.

(2) Method 2: the method based on Eqs. (26) and (28).

(3) Method 3: the modification of method 2 described in Section V.

In method 3, $\tilde{T}_1$ is estimated from Eq. (26) based on N samples, and the other $\tilde{T}_j$ are found from Eq. (28) using a later set of L samples. This is the method treated in [1].

The quantity $E(|E_2|^2 + |E_3|^2)/|\sum \tilde{w}_k \tilde{S}_k|^2$, the relative mean-square error caused by weight errors, is plotted for these three methods in Fig. 3. The formulas used are given in Appendix B. The values are plotted as a function of the signal-to-noise ratio, with the parameters L, M, and N fixed. (Throughout the curves, we take the SNR in every other channel to be one-tenth as large as the SNR in channel 1.) It is seen that method 2 for $N = L$ is better than method 3, although the difference is small at a low SNR. (For $N >> L$, method 3 would be better than method 2.) Of course, the ML value (method 1) is smallest, but the difference from method 2 is small at high SNR.

If we assume all channels have the same noise level, then method 4, the maximum-likelihood method of Section VI can also be considered. Comparing method 4 with the first three methods, it is seen that those methods can be improved by using a unified single noise estimate from the off-signal data. These improved methods are denoted by 1′, 2′, and 3′. The relative error for these four methods is shown in Fig. 4 (formulas are given in Appendix B). The behavior shown in Fig. 3 occurs here again, with the curves at a lower level, and method 4 is slightly better than method 1′.

VIII. Conclusions

Maximum-likelihood methods for combining weight estimation provide a consistent decrease in the mean-square error of the combined signal, as compared with other estimation methods, at the cost of a small increase in computational complexity. The part of the error which is caused by weight errors is decreased by over 2 dB, provided that at least as many samples are used to estimate the noise variance as the $\tilde{T}_k$. This can reduce the number of samples needed for equivalent performance to 30 percent less than the number needed by method 3.

Reference

[1] V. A. Vilnrotter and E. R. Rodemich, "A Digital Combining-Weight Estimation Algorithm for Broadband Sources With the Array Feed Compensation System," *The Telecommunications and Data Acquisition Progress Report 42-116, vol. October–December 1993*, Jet Propulsion Laboratory, Pasadena, California, pp. 86–97, February 15, 1994.

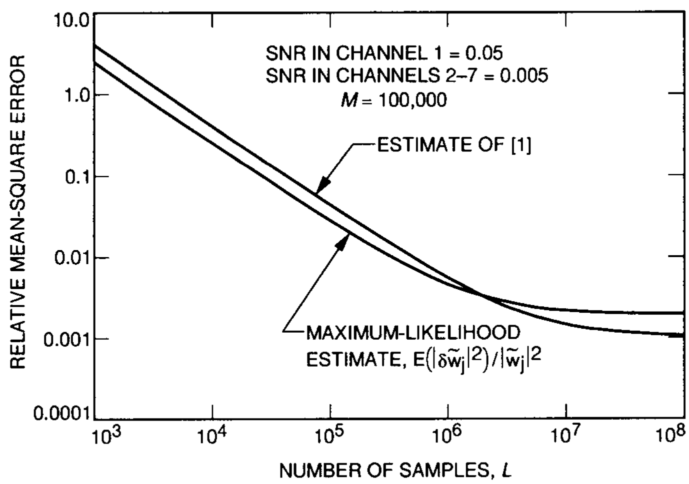

Fig. 1. Relative errors of the weight estimates.

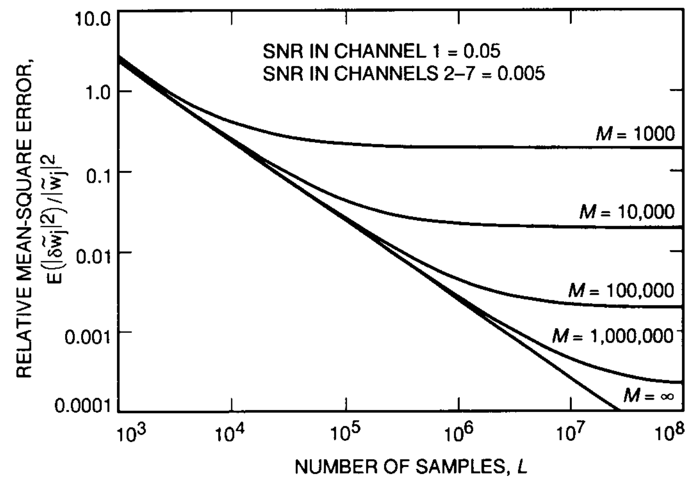

Fig. 2. Relative errors of the ML weight estimates.

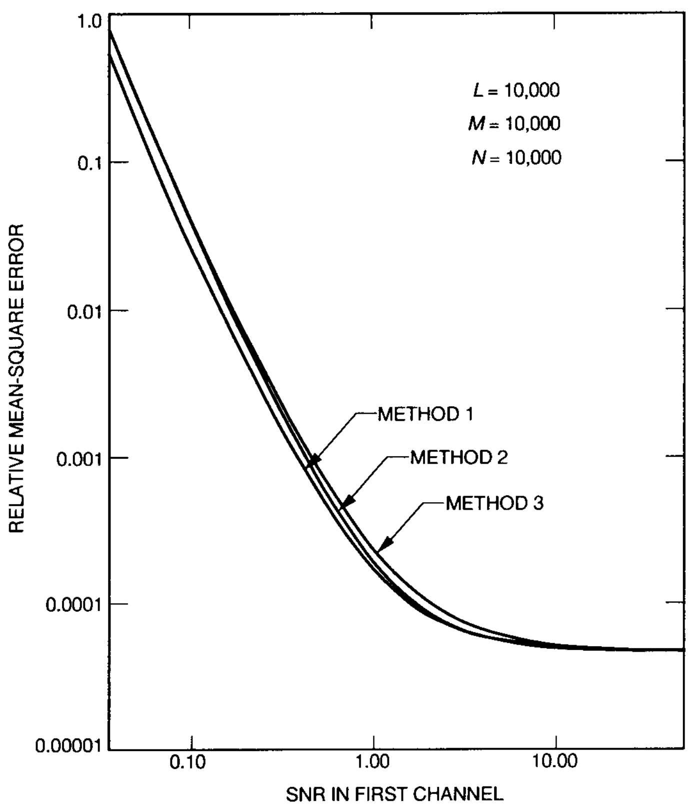

Fig. 3. Relative errors in the combined signal caused by weight errors (independent noise variances).

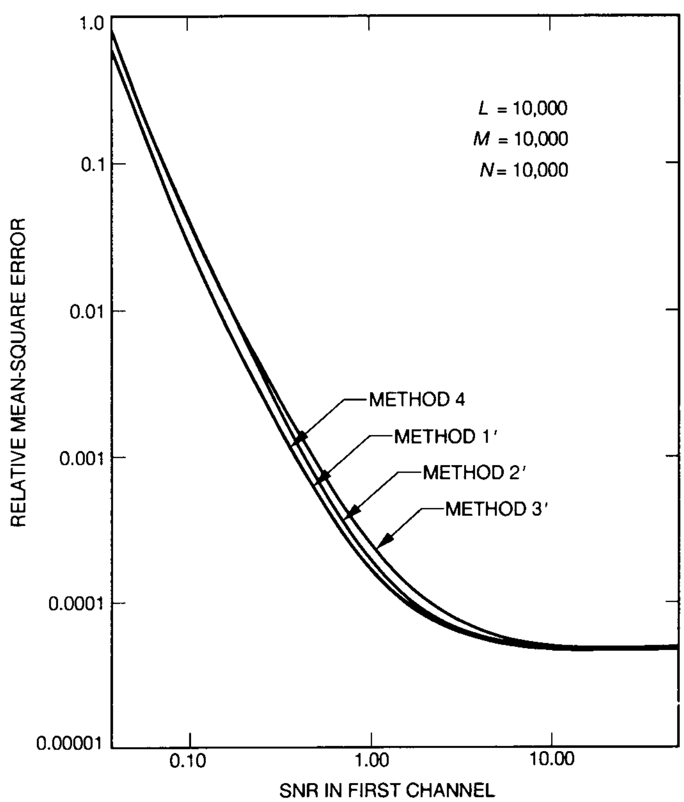

Fig. 4. Relative errors in the combined signal caused by weight errors (equal noise variances).

Appendix A

Method for Solving Eq. (13)

The mathematical problem posed by Eq. (13) is the following: Given a $K \times K$ Hermitian matrix $\tilde{\mathbf{A}}$, find its maximum eigenvalue and the corresponding eigenvector.

If the maximum eigenvalue is also the eigenvalue of maximum absolute value, then this can be accomplished by an iterative procedure. Choose any convenient starting vector $\tilde{\mathbf{x}}_0$.

If we take $\tilde{\mathbf{x}}_n = \tilde{\mathbf{A}}\tilde{\mathbf{x}}_{n-1}$ for $n \geq 1$, then $\tilde{\mathbf{x}}_n$, when normalized, approaches the eigenvector which is sought. Choose positive numbers c_n so that

$$\tilde{\mathbf{x}}_n = \frac{\tilde{\mathbf{A}}\tilde{\mathbf{x}}_{n-1}}{c_n}$$

has unit length. Then $\tilde{\mathbf{x}}_n$ approaches the eigenvector and c_n approaches the eigenvalue.

The rate of convergence of this procedure depends on the size of the next largest eigenvalue, as compared with the first. In our application, the convergence is slow for low SNR. However, the method can be modified to speed up the convergence. When suitably modified, the difference between $\tilde{\mathbf{x}}_n$ and the eigenvector decreases by a factor of the order of $(\gamma - 1)/\sqrt{L}$ at each step.

Appendix B

Formulas Used for Figs. 3 and 4

The formulas used for Figs. 3 and 4 are presented here.

For method 1,

$$E(|E_2|^2 + |E_3|^2) = \frac{\gamma(\frac{1}{4}\gamma^2 + K - 1)}{L(\gamma - 1)} + \frac{\gamma}{M(\gamma - 1)}\left[\frac{1 - 8\gamma + 4\gamma^2}{4(\gamma - 1)^2}\sum_{k=1}^{K}|\tilde{T}_k|^4 + \gamma\right] \tag{B-1}$$

For method 2,

$$E(|E_2|^2 + |E_3|^2) = \frac{1}{L}\left[\frac{\gamma(\gamma - 1)}{4|\tilde{T}_1|^4}(1 + |\tilde{T}_1|^2)^2 + (K - 1)\frac{1 + |\tilde{T}_1|^2}{|\tilde{T}_1|^2}\right]$$

$$+ \frac{1}{M}\left[\frac{\gamma(\gamma - 1)}{4|\tilde{T}_1|^4} - \frac{\gamma - 1}{|\tilde{T}_1|^2} + 2 + 2|\tilde{T}_1|^2 + \sum_{k=1}^{K}|\tilde{T}_k|^4\right] \tag{B-2}$$

For method 3,

$$E(|E_2|^2 + |E_3|^2) = \frac{1}{4}\gamma(\gamma - 1)\frac{(1 + |\tilde{T}_1|^2)^2}{N|\tilde{T}_1|^4} + \frac{1}{L}\left[\frac{K - 2 + \gamma^2}{|\tilde{T}_1|^2} + K - \gamma - 1\right]$$

$$+ \frac{1}{M}\left[\frac{\gamma(\gamma - 1)}{4|\tilde{T}_1|^4} - \frac{\gamma - 1}{|\tilde{T}_1|^2} + 2 + 2|\tilde{T}_1|^2 + \sum_{k=1}^{K}|\tilde{T}_k|^4\right] \tag{B-3}$$

For method 4,

$$E(|E_2|^2 + |E_3|^2) = \frac{\gamma}{L(\gamma - 1)}\left[\frac{1}{4}\gamma^2 + K - 1 + \frac{(\gamma - \frac{1}{2})^2}{K - 1}\right] \tag{B-4}$$

For method 1′,

$$E(|E_2|^2 + |E_3|^2) = \frac{\gamma(\frac{1}{4}\gamma^2 + K - 1)}{L(\gamma - 1)} + \frac{\gamma(\gamma - \frac{1}{2})^2}{MK(\gamma - 1)} \tag{B-5}$$

For method 2′,

$$E(|E_2|^2 + |E_3|^2) = \frac{1}{L}\left[\frac{\gamma(\gamma - 1)}{4|\tilde{T}_1|^4}(1 + |\tilde{T}_1|^2)^2 + (K - 1)\frac{1 + |\tilde{T}_1|^2}{|\tilde{T}_1|^2}\right] + \frac{1}{MK}\left[\frac{\gamma(\gamma - 1)}{4|\tilde{T}_1|^4} - \frac{\gamma^2 - 1}{|\tilde{T}_1|^2} + \gamma^2 + \gamma + 1\right] \tag{B-6}$$

For method 3′,

$$E(|E_2|^2 + |E_3|^2) = \frac{1}{4}\gamma(\gamma - 1)\frac{(1 + |\tilde{T}_1|^2)^2}{N|\tilde{T}_1|^4} + \frac{1}{L}\left[\frac{K - 2 + \gamma^2}{|\tilde{T}_1|^2} + K - \gamma - 1\right]$$

$$+ \frac{1}{MK}\left[\frac{\gamma(\gamma - 1)}{4|\tilde{T}_1|^4} - \frac{\gamma^2 - 1}{|\tilde{T}_1|^2} + \gamma^2 + \gamma + 1\right] \tag{B-7}$$